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I. BIOGRAPHY.

ARTHUR CAYLEY.

BY DR. ALEXANDER MACFARLANE.

ARTHUR CAYLEY was born at Richmond in Surrey, England, August the 16th, 1821. His father, Henry Cayley, was descended from the Cayleys of Brompton, in Yorkshire, but was at the time a merchant of St. Petersburg where he had married a Russian lady. In 1829 his parents took up their permanent residence at Blackheath in England; and Arthur was there educated at a private school for four years. At the age of 14 he was sent to King's College School, London; and the master of that school having observed the promise of a mathematical genius advised the father to educate his son not for his own business, but to enter the University of Cambridge.

In 1838 Arthur Cayley entered Trinity College, Cambridge, at the rather early age of 17. Throughout his undergraduate course he was first at his college examinations by an enormous interval, and he finished his undergraduate career in 1842 by carrying off the two highest honors, namely, the first place, or Senior Wrangler, in the Mathematical Tripos, and the first prize in the competition for the Smith Prizes. Immediately elected a Fellow of his College, he continued to reside at Cambridge for several years, during which time he lectured on mathematics, and also contributed papers to the *Cambridge Mathematical Journal*. His first contribution to that Journal was made, when he was an undergraduate, in 1841.

At that time it was necessary for a Fellow to take Holy Orders, or else resign the fellowship at the end of seven years. Mr. Cayley chose the latter alternative, and became by profession a conveyance in Lincoln's Inn, London. He followed that profession for 14 years with conspicuous ability

and success, and at the same time made many of his most important contributions to mathematical science.

About 1861 the Lucasian professorship of mathematics at Cambridge —the chair made illustrious by Sir Isaac Newton—fell vacant; it was filled by G. G. Stokes, already eminent for his work in mathematical physics, and Senior Wrangler the year before Cayley. However, it was felt desirable to secure Cayley also, and for this purpose the Sadlerian professorship of mathematics was created, which resulted in Cayley marrying and settling down at Cambridge, in 1863.

The duties of the Sadlerian professor were defined as follows: "to explain and teach the principles of pure mathematics, and to apply himself to the advancement of the science". In carrying out the former part of the duties Professsor Cayley did not give the same course of lectures year after year, but each year took for his subject that of the memoir on which he was engaged. As a consequence his students were few, for advanced work of that kind did not pay in the great mathematical examination. How well he carried out the second part of the duties may be inferred from the fact that the Royal Society Catalogue of Scientific Papers enumerates 430 memoirs contributed by him between the years 1863 and 1883, making a total up to the latter date of 724. As he continued active to the last, it is probable that the grand total of his papers does not fall short of 1000. Some of his most celebrated contributions are: Chapters in the Analytical Geometry of (n) Dimensions, On the theory of Determinants, On the theory of linear transformations, Ten Memoirs on Quantics, Memoir on the theory of Matrices, Memoirs on Skew Surfaces, otherwise Scrolls, On the Motion of Rotation of a solid Body, On the triple tangent planes of surfaces of the third order. Several of his achievements are elegantly referred to in a poem written by his colleague Clerk Maxwell in 1874, and addressed to the Committee of subscribers who had charge of the Cayley Portrait Fund:

O wretched race of men, to space confined!
What honor can ye pay to him whose mind
To that which lies beyond hath penetrated?
The symbols he hath formed shall sound his praise.
And lead him on through unimagined ways
To conquests new, in worlds not yet created.

First, ye Determinants, in order row
And massive column ranged, before him go,
To form a phalanx for his safe protection,
Ye powers of the nth root of -1!
Around his head in endless cycles run,
As unembodied spirits of direction.

And you, ye undevelopable scrolls!
Above the host wave your emblazoned rolls,
Ruled for the record of his bright inventions.
Ye cubic surfaces! by threes and nines
Draw round his camp your seven and twenty lines
The seal of Solomon in three dimensions.

March on, symbolic host! with step sublime,
Up to the flaming bounds of Space and Time!
There pause, until by Dickenson depicted,
In two dimensions, we the form may trace
Of him whose soul, too large for vulgar space,
In n dimensions flourished unrestricted.

The portrait was presented to Trinity College, and now adorns their Hall. He is represented as seated at a desk, with quill in hand, and thinking out intently some mathematical idea.

But mathematical science was advanced by Professor Cayley in yet another way. By his immense learning, his impartial judgment, and his friendly sympathy with other workers, he was eminently qualified to act as a referee on mathematical papers contributed to the various societies. Of this kind of work he did a large amount, and of his kindliness to young investigators I can speak from personal experience. Several papers which I read before the Royal Society of Edinburgh were referred to him, and he recommended their publication. Some time after I attended a meeting of the Mathematical Society of London, but the friend who would have introduced me could not be present. Professor Cayley was present, and on finding out who I was, gave me a cordial handshake, and referred in the kindest terms to the papers he had read. His was a cosmopolitan spirit, delighting only in the truth, and friendly to all seekers after the truth.

Among Cayley's papers there are several on a "Question in the Theory of Probabilities". The question was propounded by Boole, and he applied to its solution the general method of "The Laws of Thought". It was afterwards discussed by Wilbraham, Cayley and others in the *Philosophical Magazine*. My attention was drawn to the question when writing the *Principles of the Algebra of Logic*, and I ventured to contribute my idea of the question to the *Educational Times*. On mentioning the matter to Professor Kelland, he intimated pretty plainly that the discussion had been closed by Professor Cayley, and that it was temerity on my part to write anything on the subject. But the great mathematician did not think so; he wrote me a letter discussing the question and my particular way of viewing it, as well as the fundamental ideas in which I differed from Boole.

In 1882 he received a flattering invitation from the trustees of the Johns Hopkins University to deliver a course of lectures on some subject in advanced mathematics. He chose as his subject the Elliptic and Abelian functions; and the impression which his presence created has been well described by Dr. Matz in his brief notice in the January number of the MONTHLY.

Next year he was president of the British Association at the Southport meeting. In his address he spoke of the foundations of mathematics, reviewed the more important theories, traced the connection of pure with applied mathematics, and gave an outline of the vast extent of Modern Mathematics.

He regarded the complex number $a+bi$ as the fundamental quantity of mathematical analysis, and considered that with such a basis, algebra was a complete and bounded science, in which no further imaginary symbols could

spring up. It is the more remarkable that he held such a view, when we consider that early in his career he made a notable contribution to space analysis. Starting from Rodrigues' formulae for the rotation of a solid body, he arrived at the quaternion formula, and was anticipated by Hamilton only by a few months. But Cayley took a Cartesian view of analysis to the last, as is evident from the chapter which he contributed to Tait's *Treatise on Quaternions*. His aim there is to give an analytical theory of quaternions. Hamilton's aim on the other hand was to give a quaternionic theory of analysis. The difference is brought out still more strikingly in a paper printed in the last number of the *Proceedings of the Royal Society of Edinburgh.*

In 1889 the Cambridge University Press commenced the re-publication of his mathematical papers in a collected form. It was calculated that they would occupy 10 quarto volumes; 7 volumes have already appeared; and it is believed that 12 volumes will be required. No mathematician has ever had his works printed in a more handsome manner. In addition he is the author of a separate work on *Elliptic Functions.*

Space fails to enumerate the honors which he received from Universities and Scientific Academies both of the Old and of the New World. But we may mention specally, that from the Royal Society he received a Royal Medal and a Copley Medal; from the Mathematical Society of London the first De-Morgan Medal; and at the instance of the President and Members of the French Academy he was made an Officer of the Legion of Honour.

On the 26th of January he died at Cambridge. His body was laid to rest in Mill Road Cemetery in the presence of official representatives from foreign countries and many of the most illustrious philosophers of England. His spirit still speaks to us from his works, and will continue to speak to many succeeding generations.

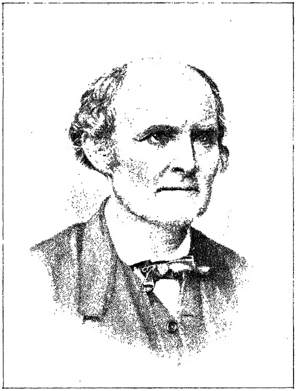
II. BIOGRAPHY.

ARTHUR CAYLEY.

BY DR. GEORGE BRUCE HALSTED.

On January 26th, 1895, after long suffering, passed away Professor Cayley, one of the very greatest masters ever known to the world of pure mathematics.

Of the great quaternion of Senior Wranglers of 1840 Leslie Ellis, 1841 Stokes, 1842 Cayley, 1843 Couch Adams, the second alone now remains.



ARTHUR CAYLEY.

Arthur Cayley was born on August 16th, 1821. His ancestor Roger de Cailli was a great lord in the reign of Henry II. His grandfather and father were both merchants in St. Petersburg. His father married a Russian, and though born in England, Cayley's mother tongue was Russian, and his features had a Russian cast. Like so many Russians, he spoke most European languages well.

Can there be anything in what has so often been cited as fact, that in the Russian race alone the brain of the woman equals that of the man in size and weight?

Arthur Cayley was a pupil of King's College School, London, and entered Trinity College, Cambridge, already a well-equipped mathematician, at the age of 17. In 1842 he took the two highest honors in the University of Cambridge, he was Senior Wrangler and First Smith's Prizeman. At that time, more than half a century ago, the Senior Wrangler was almost always as a matter of course a Johnian, so a Trinity Senior Wrangler was apt to be an object of curiosity. One of his college mates describes him at that date as a crooked little man, in no respect a beauty, and not in the least a beau. On the day of his triumph, when he was to receive his hard-earned honors in the Senate House, some of his friends combined their energies to dress him, and put him to rights properly, so that his appearance might not be altogether unworthy of his exploits and his College. He was already a man of much varied information, and that on some subjects the very opposite of scientific; for instance he was well up in all the current novels, an uncommon thing at Cambridge, where novel-reading then was not one of the popular weaknesses. His Johnian competitor for first place was a fearfully hard student, and had once worked *twenty hours a day* for a week together at a College examination. But now he almost broke down from over exertion just as the time of trial was coming on, and actually carried a supply of ether and other stimulants into the examination, in case of accidents. Nevertheless he made a good fight of it, and having great *pace* as well as *style* in addition to his knowledge, beat Cayley a little on the bookwork, but was beaten two hundred marks in problems, which decided the contest.

One of the low bookwork papers to which three hours were allotted happening to be rather shorter than usual, the man from St. John's, either as a bit of bravado to frighten his opponent, or because having done all that could be done he had no reason for waiting longer, came out at the expiration of two hours, having floored the paper in that time. His early exit did not escape notice, and the same evening a Trinity Senior Soph rushed up in great fear to the room of his friend, on whom the hopes of the College depended. "Cayley! Cayley! they tell me S— floored the paper this afternoon in two hours. Is it so?" The mathematician, who was refreshing himself after the fatigues of the day with the innocent and economical luxury of a footbath, looked up at the querist from his tub with the equanimity of a Diogenes, and replied: "Likely enough he did. I floored it myself in two hours and a half". The examination for the Smith's Prizes which took place immediatly after the

result of the Mathematical Tripos was declared, had a similar result: Cayley beat his opponent, but with nothing to spare. The matter was very different the next year, when Couch Adams, the discoverer of Neptune, won not only easily, but had three thousand marks to the Second Wrangler's fourteen hundred, so that there was more numerical difference between them than between the Second Wrangler and the *spoon*, or last man. But this was produced by a singular case of fright or stampede which occurred at this examination. The man who would have been second, like Adams a Johnian, took fright when four of the six days were over, and actually ran away, not only from the examination but out of Cambridge, and was not discovered by his friends or family till some time after. Even as it was, and without the last two days, he came out ninth in the list of wranglers. But even if Cayley had been beaten for first place, he might still have been equally as eminent as now; for has not Cambridge that other tremendous tetrad, Sylvester, Wm. Thomson, Clerk-Maxwell, Clifford, all Second Wranglers!

In 1841 Cayley published his first paper, thus commencing the astounding series of over 800 memoirs with which he so enriched his science. The collected edition of his works now being published by the University Press will extend to ten or more quarto volumes, a scientific monument equally unique in amount, range, and quality.

After his election to a Fellowship, which, as he was unwilling to take Holy Orders, could be only temporary, he studied conveyancing in London, and at Lincoln's Inn first met his greatest and lifelong friend and fellow-genius Sylvester, for they had never met at Cambridge, where Sylvester was Second Wrangler in 1837.

He practised as a conveyancer for 14 years, but during this time his real occupation was pure mathematics, and in those years some of his most notable discoveries were made. The law was always drudgery to him. The superabundant verbiage of legal forms was always distasteful to him. He once remarked that "the object of law was to say a thing in the greatest number of words, of mathematics to say it in the fewest."

Cayley was a very gentle, sweet character. Sylvester told me that he never saw him angry but once, and that was when a messenger broke in on one of their interviews with a mass of legal documents, new business for Cayley. In an access of disgust, Cayley dashed the documents upon the floor.

In 1863 Lady Sadler's various trusts were consolidated, and a new Sadlerian Professorship of Pure Mathematics was created in the University of Cambridge, especially for Cayley. As chairman of the Association for Promoting the Higher Education of Women he did most to raise Newnham College to its present influential position.

In Cambridge he was accustomed to give the small classes of advanced students who were prepared to follow him no mere routine course, but, like the best German professors since Jacobi, the latest and highest work on which he was at the time engaged.

As early as 1852 he was a fellow of the Royal Society. In 1858 he

joined Sylvester and Stokes in starting the Quarterly Journal of Pure and Applied Mathematics. In 1882 he delivered a special course of lectures at the Johns Hopkins University, where Sylvester was still professor. Baltimore was then the apex not only of the Western Continent, but of the world, for Salmon soon after said that if European mathematicians had to elect themselves a head, it would be Cayley. In 1863 he married and settled permanently in Cambridge.

Cayley was assuredly the most learned and erudite of mathematicians. Of him it might be said, he knew everything, and he was the very last man who ever will know everything. I have heard Sylvester say that when he wished to know anything he simply asked Cayley, for to Sylvester it was not only often irksome to study what had been done by others, but impossible, since the very beginning of such study was sure to start in him a train of original thought and research which absorbed him irresistably. This wideness of knowledge made Cayley invaluable as a mathematical referee. To the Royal Society, the Mathematical Society, the Royal Astronomical Society, the Cambridge Philosophical Society he was long the principal adviser as to the merits of mathematical papers presented for publication. Cayley's erudition gave his originality always the most fertile fields.

In 1841 the wonderful George Boole, the creator of algorithmic logic, made use of a simple case of what we would now call *invariance* in linear substitutions. Then Cayley set himself the problem to determine *a priori* what functions of the coefficients of a given equation possess this property. He called such functions hyperdeterminants, until Sylvester the mathematical Adam, who names the creatures, called them *invariants*. Substitutions and invariance are now the heart of the very latest analytic mathematics, and have received an extraordinary transformation and development at the hands of Sophus Lie.

Again, the idea that any metrical property in geometry could be looked upon as a projective relation in a particular configuration began to occur in the French school. For example Laguerre in 1853 so expresses an angle. But in 1859 in his sixth memoir on Quantics Cayley published his solution of the general problem he had set himself of finding a general theory of projective metrics of which ordinary metrics should be a special case; thus breaking down the distinction between pure positional or descriptive geometry and the ordinary metrical geometry by merging all into projective geometry.

Remembering that von Staudt had founded cross-ratio on a pure projective basis in his theory of the *Wurf*, entirely without using measurement in the ordinary sense (direct comparison as to size by congruence), Klein saw that Cayley's theory of projective measurement leads directly to the three possible cases of geometry, Euclidean and non-Euclidean, which he called parabolic, elliptic, hyperbolic. The hyperbolic is the now well-known non-Euclidean geometry of Lobachevsky and Bolyai. Thus Cayley's doctrine of "the absolute", already greatly admired, was given additional importance, and its creation will ever rank as one of the very greatest of his achievements.

As a third epoch-making production of his fertile and tireless genius we may mention the theory of matrices, on which multiple algebra is based. In this, as in the theory of invariants, Sylvester was his most brilliant coadjutor. Cayley was a devoted admirer of Euclid. In his great address as President of the British Association, speaking of Greek mathematics he says: "But the earliest extant writings are those of Euclid (B. C. 285).

There is hardly anything in mathematics more beautiful than his wondrous fifth book: and he has also in the seventh, eighth, ninth, and tenth books fully and ably developed the first principles of the theory of numbers, including the theory of incommensurables". In the same address he says: "It is well known that Euclid's twelfth axiom, even in Playfair's form of it, has been considered as needing demonstration; and that Lobachevsky constructed a perfectly consistent theory, wherein this axiom was assumed not to hold good, or say a system of non-Euclidean plane geometry. There is a like system of non-Euclidean solid geometry. Riemann's view was that having *in intellectu* a more general notion of space (in fact a notion of non-Euclidean space), we learn by experience that space (the physical space of our experience), is, if not exactly, at least approximately, Euclidean space. But suppose the physical space of our experience to be thus only approximately Euclidean space, what is the consequence which follows? Not that the propositions of geometry are only approximately true, but that they remain absolutely true in regard to that Euclidean space which has been so long regarded as being the physical space of our experience.

The three geometries (spherical, Euclidean, and Lobachevsky's) should be regarded as members of a system—viz., they are the geometries of a plane (two-dimensional) space of constant positive curvature, zero curvature, and constant negative curvature respectively; or again they are the plane geometries corresponding to three different notions of distance; in this point of view they are Klein's elliptic, parabolic, and hyperbolic geometries respectively."

But here this imperfect sketch must stop. Enough that his life furthered in the highest degree the aim of his university, in the words of his mother's compatriot Lobachevsky, "not only to enlighten the spirit with knowledge, but also to inculcate virtues, to implant a desire for glory, a feeling of nobility, justice, and honor, of strict and sacred honesty, that would resist all cases of temptation, apart from any fear of punishment."

SOME NOTES ON THE THEORY OF PROBABILITY.

By Professor G. B. M. ZERR. A. M., Principal of High School, Staunton, Virginia.

Of the three memoirs written by Prevost and Lhuilier, the first entitled, "Sur les Probabilites," was read November 12th, 1795 and occupies pages 117-142 of the mathematical portion of the volume, "Memoires de l'Acad. .... Berlin. This important Memoir deals with the following venerable problem:

An urn contains m balls some of which are white and the rest black, but the number of each is unknown. Suppose p white balls and q black balls have been drawn and not replaced; required the probability that out of the next $r+s$ drawings r shall give white balls and s black balls.

This problem suggests the discussion of the more general problem: From an unknown number of balls each equally likely to be any of n colors $a_1+a_2+a_3+\ \ldots.\ +a_n$ balls are drawn and turn out a_1 of the first color, a_2 of the second, a_3 of the third,$\ldots.a_n$ of the nth. If $b_1+b_2+b_3+\ldots.+b_n$ more balls are drawn find the probability that b_1 are of the first color, b_2 of the second, $\ldots.$, b_n of the nth. The problem will not be altered, if we suppose the balls arranges along a straight line of length unity on n differant portions of the line. Call the first portion x_{n-1}, the sum of the first and second, x_{n-2}, the sum of the first three, $x_{n-3},\ldots$ the sum of the first $(n-1)$, x_1, then we get for the required chance the following definite integral:

$$p=\frac{\underline{|b_1+b_2+b_3+\ldots.+b_n}}{\underline{|b_1}\,\underline{|b_2}\,\underline{|b_3}\ldots.\underline{|b_n}}\;\frac{\int_0^1\int_0^{x_1}\int_0^{x_2}\ldots.\int_0^{x_{n-2}}(1-x_1)^{a_1+b_1}(x_1-x_2)^{a_2+b_2}}{\int_0^1\int_0^{x_1}\int_0^{x_2}\ldots.\int_0^{x_{n-2}}(1-x_1)^{a_1}(x_1-x_2)^{a_2}}$$

$$\frac{(x_2-x_3)^{a_3+b_3}\ldots.x_{n-1}^{a_n+b_n}dx_1dx_2dx_3\ldots.dx_{n-1}}{(x_2-x_3)^{a_3}\ldots.x_{n-1}^{a_n}dx_1dx_2dx_3\ldots.dx_{n-1}}$$

$$=\frac{\underline{|b_1+b_2+b_3+\ldots.+b_n}\;\underline{|a_1+b_1}\;\underline{|a_2+b_2}\;\underline{|a_3+b_3}}{\underline{|b_1}\;\underline{|b_2}\;\underline{|b_3}\ldots.\underline{|b_n}\;\underline{|a_1}\;\underline{|a_2}\;\underline{|a_3}\ldots.\underline{|a_n}}$$

$$\frac{\ldots.\underline{|a_n+b_n}\;\underline{|a_1+a_2+a_3+\ldots.+a_n+n-1}}{\underline{|a_1+a_2+a_3+\ldots.+a_n+b_1+b_2+b_3+\ldots.+b_n+n-1}}.$$

When $n=2$,

$$p=\frac{\underline{|b_1+b_2}\;\underline{|a_1+b_1}\;\underline{|a_2+b_2}\;\underline{|a_1+a_2+1}}{\underline{|b_1}\;\underline{|b_2}\;\underline{|a_1}\;\underline{|a_2}\;\underline{|a_1+a_2+b_1+b_2+1}},$$ same result as given in the memoir above referred to.

When $n=3$,

$$p=\frac{\underline{|b_1+b_2+b_3}\;\underline{|a_1+b_1}\;\underline{|a_2+b_2}\;\underline{|a_3+b_3}\;\underline{|a_1+a_2+a_3+2}}{\underline{|b_1}\;\underline{|b_2}\;\underline{|b_3}\;\underline{|a_1}\;\underline{|a_2}\;\underline{|a_3}\;\underline{|a_1+a_2+a_3+b_1+b_2+b_3+2}}$$

Let $\dfrac{\underline{|a_1+a_2+a_3+2}}{\underline{|a_1+a_2+a_3+b_1+b_2+b_3+2}}=A$, then the chance that all are of the first color is $p_1=\dfrac{\underline{|a_1+b_1+b_2+b_3}}{\underline{|a_1}}A$. $p_2=\dfrac{\underline{|a_2+b_1+b_2+b_3}}{\underline{|a_2}}A$ that all are of the second color. $p_3=\dfrac{\underline{|a_3+b_1+b_2+b_3}}{\underline{|a_3}}A$. that all are of the third color.

$p_4=\dfrac{\underline{|b_1+b_2+b_3}\;\underline{|a_1+b_1+b_2}\;\underline{|a_3+b_3}}{\underline{|b_1+b_2}\;\underline{|b_3}\;\underline{|a_1}\;\underline{|a_3}}A$, that b_1+b_2 are of the first color and b_3 of the third.

$p_5=\dfrac{\underline{|b_1+b_2+b_3}\;\underline{|a_1+b_3}\;\underline{|a_3+b_1+b_2}}{\underline{|b_1+b_2}\;\underline{|b_3}\;\underline{|a_1}\;\underline{|a_3}}A$, that b_3 are of the first color and b_1+b_2 of the third and so on for any combination.

When $a_1=5$, $a_2=3$, $a_3=2$, $b_1=b_2=b_3=1$.

Then $p=\frac{72}{455}$, $p_1=\frac{8}{65}$, $p_2=\frac{4}{91}$, $p_3=\frac{2}{91}$, $p_4=\frac{9}{65}$, $p_5=\frac{36}{455}$. The chance that there are none of the second color is $p_6=p_1+p_3+p_4+p_5=\frac{33}{91}$.

These numerical results are the same as those obtained by an algebraic solution of the same problem given by the late Professor Wolstenholme.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED. A. M., (Princeton); Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the March Number.]

PROPOSITION XVI. *By any quadrilateral $ABCD$, of which the four angles together are equal to, or greater, or less than four right angles, is established respectively the hypothesis of right angle, or obtuse angle, or acute angle.*

Proof. Join AC. The three angles of the triangle ABC (fig. 14.) will not be together equal to, or greater, or less than two right angles, without the three angles of the triangle ADC being themselves also together respectively equal to, or greater, or less than two right angles, lest obviously (by the preceding) from one of those triangles be established one hypothesis, and another from the other, against the fifth, sixth, and seventh propositions of this work.

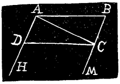


Fig. 14.

This holding good: If the four angles together of the premised quadrilateral are equal to four right angles, it follows that the three angles together of either of the just mentioned triangles will be equal to two right angles, and therefore (from the preceding) the hypothesis of right angle will be established.

But if indeed the four angles of this quadrilateral be together greater, or less than four right angles, similarly the three angles together of those triangles should be respectively either at the same time greater, or at the same time less than two right angles. Wherefore from these triangles would be established respectively (from the preceding) either the hypothesis of obtuse angle, or the hypothesis of acute angle.

Therefore by any quadrilateral, of which the four angles together are equal to, or greater, or less than four right angles, is established respectively the hypothesis of right angle, or obtuse angle, or acute angle. Quod erat demonstrandum.

COROLLARY. Hence: any two opposite sides of the premised quadrilateral being produced toward the same parts, as suppose AD to H, and BC to M; the two external angles HDC, MCD will be (Eu. I. 13.) either equal to, or less, or greater than the two internal and opposite angles together at the points A, and B, according as is true the hypothesis of right angle, or obtuse angle, or acute angle.

[To be continued.]

SOME SUGGESTIONS ABOUT VARIATION.

By ERIC DOOLITTLE, Professor of Mathematics, State University of Iowa, Iowa City, Iowa.

I think that in all of our Algebras the fundamental definitions of variation being in the idea of the ratio of two values of two changing quantities; the questions are first stated in the form of Proportions, and finally reduced to simple equations for solution.

It seems to me that this method is more cumbersome than necessary, especially in regard to the fundamental definitions, which thus involve such complex ideas that the student has a good deal of difficulty to grasp and apply

them; at least that has been my experience with young students. I have therefore given the definitions in the form of equations in the first place, which seems to considerably simplify the theorems and their application.

Assuming the student to know what variables and constants are, the presentation of the subject might be somewhat as follows.

(1). If $x,y,z,w....$ are varying quantities, then v is said to be a *function* of $x,y,z,w....$ when any change in the value of any or all of these variables produces a change in the value of v. Thus v will vary when $x,y,z,w....$ vary.

(2). y is said to vary *as* x, if y always equals m times x, where m is constant, whatever be the value of x. This is the simplest kind of variation,* and is sometimes expressed by saying "y varies directly as x." Thus, if a train go m miles an hour, the distance (y) varies directly as the number of hours (x), since $y=mx$.

(3). y is said to vary *inversely* as x, if y always equals m times the inverse of x; that is, if $y=m(1/x)$. Thus in the last illustration, if it require x hours for a train to go m miles, then the speed in miles per hour (y) varies inversely as x, since $y=\frac{m}{x}$.

(4). y is said to vary directly as $x,u,z,w....$ if y always equals m times the product $xuzw...$ where, as before, m is constant.

(5). Finally, y may be said to vary directly as certain quantities, and inversely as certain others, if y always equals m times the continued product of the former and the inverse of each of the latter. Thus y varies directly as z^2 and $a+x$, and inversely as $\beta^3 x$ if $y=mxz^2\times(a+x)\times\frac{1}{\beta^3 x}$.

(6). The equations arising from the last four definitions may be called the *Statement* of the variation, and the first step toward the solution of any problem in variation is to write this statement. We then substitute in it such values as are known, and solve for what is required. If there are several different conditions in the problem, we make the statement for each separately, and solve the resulting simultaneous equations.

Numerous examples and illustrations of these principles should of course be given, and the proof of the more elementary theorems should follow. It will be seen that they are almost self evident by this method of treating the subject.

For instance, "If y varies as x, and x varies as z, then y varies as z";

"If y varies as x, and y' varies as x', then yy' varies as xx'";

"If y varies as zx, then x varies as y/z, and z varies as y/x" etc.

"If y is a function of two variables only, x and z; and if y varies as x when z is constant but when x is constant y varies as z, then y varies with x and z at once; that is $y=mxz$."

If the above definitions be admitted, the following proof of the last

* The distinction between variation in general and the simplest possible kind of it, is here introduced to guard against the supposition that all variation is of this simplest possible kind. This danger is pointed out by Dr. Chrystal on page 275 part 1 of his Algebra.

theorem may take the place of the longer one usually given:

"Since y varies with x, multiplying x alone will multiply y by the same factor; and similarly, multiplying z alone will multiply y; hence multiplying both x and z will twice multiply y, once by each of the respective factors. Hence x and z must enter as factors of the value of y, and since there are no other variable factors, y equals a constant expression times xz. That is $y=mxz$."

I suppose this subject of variation is pretty generally omitted by Preparatory and High School classes in Algebra. It seems to me to furnish an excellent opportunity to emphasize the difference between constant and variable quantities; a distinction the student usually meets here for the first time. I think it repays a few days careful work, by the introduction it thus gives to Analytic Geometry and the Calculus. Besides, by a few obvious applications to Astronomy and Physics, it can be made of interest to the pupil.

Such an oasis, after travelling in the desert of Radicals and Imaginaries, is very welcome.

TRUE PROPOSITIONS NOT INVALIDATED BY DEFECTIVE PROOFS.

By Professor John N. Lyle, Ph. D., Westminster College, Fulton, Missouri.

A bad cause may be brilliantly advocated and a good one poorly defended. A false proposition may be supported by plausible arguments and a true one by defective and even erroneous proofs. The true proposition is not thereby shown to be false or unworthy of acceptance.

John Playfair's demonstration of the angle-sum of a rectilineal triangle may be unsatisfactory and yet the proposition that the angle-sum is two right angles may be rigorously true and its contradictory absolutely false.

Legendre's demonstration that the angle-sum can not be less than two right angles is said by Professor Halsted to be "disgraceful." Even if this be admitted, it does not follow that the proposition itself should be doubted or rejected.

Discrediting Legendre's demonstration furnishes no legitimate warrant for postulating the truth of the hypothesis that the angle-sum can be less than two right angles.

The proofs that the angle-sum can be neither greater nor less than two right angles given in the pamphlet—Euclid and the Anti-Euclidians—may fall below the standard required by rigid geometrical science, but this does not justify the acceptance as true of the assumption that the angle-sum is greater or less than two right angles.

Lobatschewsky's theorem that the angle sum can not be greater than

two right angles is manifestly in conflict with the doctrine of those metageometers who maintain that *the space in which we dwell* has constant, positive curvature and that the angle-sum of the rectilineal triangle drawn therein is greater than two right angles.

If it is maintained that the conclusions of Lobatschewsky, Riemann and Euclid are consistent with their respective premises, the question arises which of these systems is true. If any one does not really know which is right, confession of one's ignorance may be good for the soul, but can hardly be received as satisfactory evidence that the agnostic is in possession of geometrical science.

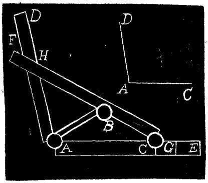
The hypothesis that Lobatschewsky, Euclid and Riemann all three tell the truth is confronted with the difficulty that they contradict each other.

Professor Halsted teaches as sound geometry the views of each of these three writers. I can not accept this teaching. If the Euclidian doctrine is true, according to logical law that which contradicts it must be false. This procedure of Professor Halsted antagonized the logical laws of non-contradiction and excluded Middle whether he is aware of it or not.

A TRISECTOR OF ANGLES.

By M. A. GRUBER, A. M., War Department, Washington, D. C.

Description. A, B, and C are centers and joints. G is a slide moving along the rule AE. The joint C is fixed to the slide so that the center C moves in the line AC. FC is a rule finely and accurately graduated from B to F, and fixed to the slide G by the joint C. AD is a fine and accurately graduated rule fixed to the rule AE by the joint A. AB is a small rule jointed at A and B.



Line AB equals line BC, both remaining constant.

The *edges* of the rules for use are those radiating from the centers.

Use. It is desired to trisect the $\angle DAC$.

Place the center A of the trisector upon the vertex A of the angle, so that the edge AC of the rule AE coincides with the side AC of the angle. Then move the rule AD until the edge coincides with the side AD of the angle. Now move the slide G until BH on the rule FC equals AH on the rule AD. Then draw a line along edge of rule AB.

$\angle BAC = \frac{1}{3} \angle DAC$. Bisect $\angle DAB$ and the trisection is complete.

Proof. $BC=AB$ and $BH=AH$ by construction.

$\angle HBA=\angle BAC+\angle ACA=2\angle BAC$. But $\angle HBA=\angle HAB$.

$\therefore \quad \angle HAC=\angle HAB+\angle BAC=3\angle BAC$.

Within reasonable limits of length of the rules FC and AD, angles up to 120 can be trisected.

History. Last February four years ago, I was experimenting with triangles. I had drawn a rt.△ whose acute angles were 60° and 30°. By joining the vertex of the rt.∠ with the middle of the hypothenuse, I noticed that the rt.∠ was trisected. To devise an instrument for the trisection of any angle then engaged my mind for a few weeks, and the above device was the result.

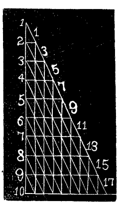
I communicated my discovery to several mathematicians and inquired as to its practicability. The replies were not encouraging. One reason given was that an instrument with several joints and a slide, was not sufficiently accurate. The suggestion was also made that it would not pay to get it patented, as the trisecting of angles entered to a very limited extent in the mechanical applications.

Thinking that the readers of the AMERICAN MATHEMATICAL MONTHLY might be interested in this device, though it may be but a mathematical curiosity, I have given the foregoing brief sketch of it.

DIAGRAM FOR THE LAWS OF THE FALLING BODIES.

By Rev. A. L. GRIDLEY, Pastor of the Congregational Church, Kidder, Missouri.

Let the distance a body would fall in one second be represented by one of the small triangles in diagram as a. During the first second it would fall through the firstspace, or triangle at the apex. During the second second it would pass through three, as that is the number of triangles in the second space which is indicated by the figures at the right. During the two second it would pass through $3+1$ triangles$=4a$, or $2^2\times a$.



To illustrate farther. How far would a body fall during the 9thsecond of its descent?

Opposite the figure 9 on the left are 17 triangles so it would pass through 17 times the distance it did during the first second or $17a$. How far would it fall during the ninth second without increment?

Leave off the right hand triangle and there would remain 18 so it would fall $18a$.

What would be the velocity at, say, the end of the 8th second? It would be the distance it would fall during the 9th second without increment, or the triangle at right hand side,$=16a$.

How far would it fall in, say, 9 seconds. Of course $9^2\,a$ or the sum of all the triangles in the first nine spaces.

With what velocity must a body be projected upward in order to rise during 10 seconds? Opposite 10 are 19 triangels so the initial velocity should be $19a$.

By a little thought any rule or problem in falling bodies can be *counted* out upon the diagram and it is unnecessary to commit any rule to memory as it can be produced at any moment from the diagram. Even the recollection will usually be sufficient to solve an ordinary problem as it has done with the inventor of the diagram—the writer—for thirty-five or forty years.

ERRATUM.—Owing to the extravagance of the compositor a needless *the* was inserted in the title of this paper.—PUBLISHERS.

ARITHMETIC.

Conducted by B.F.FINKEL, Kidder, Missouri. All Contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

38. Proposed by J. A. CALDERHEAD, B. Sc., Superintendent of Schools, Limaville, Ohio.

What must be the thickness of a 36-inch shell, in order that it may weigh 1 ton, supposing a 13-inch shell to weigh 200 pounds, when two inches thick.

IV. Solution by the PROPOSER.

$$200:2000::13^3-9^3:36^3-r^3;\text{ whence } r=31.74 \text{ inches.}$$

$$\therefore\quad (36-31.74)\div2=2.13 \text{ inches}=\text{thickness of 36-inch shell.}$$

39. Proposed by P. C. CULLEN, Superintendent of Schools, Brady, Nebraska.

A, *B*, and *C* start from same point at same time. *A* north at rate of three miles per hour, *B* east at rate of four miles and *C* west at rate of five miles per hour. *B* at end of two hours starts at such an angle as to intersect *A*. How long after starting must *C* start north-west in order to meet *A* and *B* at common point?

II. Solution by Professor H. W. DRAUGHON, Olio, Mississippi.

While *B* travels 8 miles east, *A* travels 6 miles north. The rest of *A*'s distance north, and the distance *B* travels after turning, are in the ratio of 3 to 4. Since *B*'s latter distance is on the hypothenuse of a right triangle, whose base is 8 miles and perpendicular, *A*'s distance, we have from Geometry, (hypothenuse$+8$)(hypothenuse-8)$=(\frac{3}{4}$hypothenuse$+6)^2=\frac{9}{16}$(hypothenuse$+8)^2$: whence, by division, we get hypothenuse$-8=\frac{9}{16}$(hypothenuse$+8$). $\therefore$ hypothenuse$=28\frac{4}{7}$ miles; and the perpendicular, $=$ *A*'s distance north,

$=\sqrt{(28\frac{4}{7})^2-8^2}=27\frac{3}{7}$ miles. Now C's route forms with A's route a right triangle whose perpendicular is $27\frac{3}{7}$. The sum of the hypothenuse and base$=C$'s distance$=\frac{5}{3}$ of A's distance$=27\frac{3}{7}\times\frac{5}{3}=45\frac{5}{7}$ miles. Also, from Geometry, the difference between the hypothenuse and base$=(27\frac{3}{7})^2\div45\frac{5}{7}=16\frac{16}{35}$ miles.

$\therefore$ Base$=\frac{1}{2}(45\frac{5}{7}-16\frac{16}{35})=14\frac{22}{35}$ miles. C's time in base is therefore, $14\frac{22}{35}\div5=2\frac{162}{175}$ hours$=2$ hours 55 minutes $32\frac{4}{7}$ seconds.

42. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

If $m=2$ct. be the interest on $M=100$ct. for $p=40$ days, find the yearly rate per cent.

I. Solution by P. S. BERG, Apple Creek, Ohio.

$\frac{1}{9}$ cent$=$the interest on 100 cents for 40 days at 1%.

$2\div\frac{1}{9}=18$. Hence, 18 is the yearly rate per cent.

II. Solution by COOPER D. SCHMITT, Professor of Mathematics, Vanderbilt University, Knoxville, Tennessee.

If m is the interest on M cents for p days, then $\dfrac{m}{p}$ is the interest on M cents for 1 day, and $\dfrac{360m}{p}$ is the interest on M cents for 360 days.

Hence the per cent will be $\dfrac{360m}{p}$ of $100=\dfrac{36000m}{Mp}$ %. If $m=2, p=40$, and $M=100$, the rate is $\dfrac{72000}{100\times40}=18\%$.

Solutions of this problem were received from Professors *Matz* and *Zerr*.

43. Proposed by B. F. BURLESON, Oneida Castle, New York.

A, in a scuffle, seized on $\frac{2}{3}$ of a parcel of sugar plums; B caught $\frac{3}{8}$ of it out of his hands, and C laid hold on $\frac{3}{10}$ more: D ran off with all A had left, except $\frac{1}{7}$ which E afterwards secured slyly for himself; then A and C jointly set upon B, who, in the conflict, let fall $\frac{1}{2}$ he had, which were equally picked up by D and E, who lay perdu. B then kicked down C's hat, and to work they all went anew, for what it contained; of which, A got $\frac{1}{4}$, B $\frac{1}{3}$, and D $\frac{2}{7}$, and C and E equal shares of what was left of that stock. D then stuck $\frac{3}{4}$ of what A and B last acquired, out of their hands; they, with difficulty, recovered $\frac{5}{8}$ of it in equal shares again, but the other three carried off $\frac{1}{8}$ apiece of the same. Upon this, they called a truce, and agreed that the $\frac{1}{3}$ of the whole, left by A at first, should be equally divided among them. How much of the prize, after this distribution, remained with each of the competitors?

I. Solution by A. L. FOOTE, C. E., Middleburg, Connecticut.

First, A has $\frac{2}{3}$; second, A has $\frac{2}{3}-(\frac{3}{8}+\frac{3}{10})$ of $\frac{2}{3}=\frac{13}{60}$, B has $\frac{3}{8}$ of $\frac{2}{3}=\frac{1}{4}$, and C has $\frac{3}{10}$ of $\frac{2}{3}=\frac{1}{5}$; third, A has $\frac{13}{60}-(\frac{13}{70}+\frac{13}{420})=0$, B has $\frac{1}{4}$, C, $\frac{1}{5}$, D $\frac{6}{7}$ of $\frac{13}{60}=\frac{13}{70}$, and E, $\frac{1}{7}$ of $\frac{13}{60}=\frac{13}{420}$; fourth, A has 0, B has $\frac{1}{2}$ of $\frac{1}{4}=\frac{1}{8}$, C has $\frac{1}{5}$, D has $\frac{13}{70}+\frac{1}{16}=\frac{139}{560}$, and E has $\frac{13}{420}+\frac{1}{16}=\frac{157}{1680}$; fifth, A has $\frac{1}{4}$ of $\frac{1}{5}=\frac{1}{20}$, B has $\frac{1}{8}+(\frac{1}{3}$ of $\frac{1}{5})=\frac{23}{120}$, C has $\frac{1}{5}-(\frac{1}{20}+\frac{1}{15}+\frac{2}{35})=\frac{11}{420}$, D has $\frac{139}{560}+(\frac{2}{7}$ of $\frac{1}{5})=\frac{171}{560}$, and E has $\frac{157}{1680}$; sixth, A has $\frac{1}{20}-(\frac{3}{4}$ of $\frac{1}{20})=\frac{1}{80}$, B has $\frac{23}{120}-(\frac{3}{4}$ of $\frac{1}{15})=\frac{17}{120}$, C has $\frac{11}{420}$, D has $\frac{171}{560}+\frac{7}{80}=\frac{11}{28}$, and E has $\frac{157}{1680}$; seventh, A has $\frac{1}{80}+\frac{7}{256}=\frac{51}{1280}$, B has

$\frac{17}{120}+\frac{7}{256}=\frac{649}{3840}$, C has $\frac{11}{420}$, D has $\frac{11}{28}-\frac{7}{128}=\frac{303}{896}$, and E has $\frac{157}{1680}$; eight, A has $\frac{51}{1280}-\frac{21}{2048}+\frac{1}{15}=\frac{2957}{30720}$, B has $\frac{649}{3840}-\frac{21}{2048}+\frac{1}{15}=\frac{6925}{30720}$, C has $\frac{11}{420}+\frac{7}{1024}+\frac{1}{15}=\frac{10719}{107520}$, D has $\frac{303}{896}+\frac{7}{1024}+\frac{1}{15}=\frac{44263}{107520}$, and E has $\frac{157}{1680}+\frac{7}{1024}+\frac{1}{15}=\frac{17951}{107520}$, or reducing these fractions to a common denominator, we have the following: A $\frac{20699}{215040}$, B $\frac{48475}{215040}$, C $\frac{21438}{215040}$, D $\frac{88526}{215040}$, E $\frac{35902}{215040}$, the sum of which is $\frac{215040}{215040}=1$ as it should be.

Excellent solutions of this problem were received from *G. B. M. Zerr*, *E. W. Morrell*, and *P. S. Berg*

ERRATUM—In the solution of problem 42, Professor Cooper D. Schmitt's address should read, Professor of Mathematics, University of Tenn. etc.

PROBLEMS.

48. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Fifty thousand days preceding Thursday, March 7, 1895, was what date and what day of the week?

49. Proposed by J. A. CALDERHEAD, B. Sc., Superintendent of Schools, Limaville, Ohio.

I have a garden in the form of an equilateral triangle, whose sides are 200 feet. At each corner stands a tower; the height of the first is 30 feet, the second is 40 feet, and the third is 50 feet. At what distance from the base of each tower must a ladder be placed, that it may just reach the top of each? And what is the length of the ladder, the garden being a horizontal plane?

[From *Greenleaf's National Arithmetic.*]

Give a solution simple enough to be presented to a class in arithmetic.

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

39. Proposed by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Find x, y, z, and w from the equations

$$x^4+y^4+z^4+x^2+y^2+z^2=112\ldots.(1),$$
$$x^4+z^4+w^4+x^2+z^2+w^2=382\ldots.(2),$$
$$x^4+y^4+w^4+x^2+y^2+w^2=294\ldots.(3),$$
$$y^4+z^4+w^4+y^2+z^2+w^2=364\ldots.(4).$$

I. Solution by A. H. BELL, Hillsboro, Illinois, P. S. BERG, Apple Creek, Ohio, D. G. DURRANCE, Jr., Camden, N.Y., COOPER D. SCHMITT, A.M., University of Tennessee, and H.C.WILKES, Murraysville, West Virginia.

Adding the four equations and dividing the result by 3, we readily obtain, $x^2+y^2+z^2+w^2+x^4+y^4+z^4+w^4=384$. From this subtract each equation in order and we obtain $w^2+w^4=272$, $y^2+y^4=2$, $z^2+z^4=90$, $x^2+x^4=20$, all of which are bi-quadratics. Solving we find, $w=\pm4$ or $\pm\sqrt{-17}$, $y=\pm1$ or $\pm\sqrt{-2}$, $z=\pm3$ or $\pm\sqrt{-10}$, $x=\pm2$ or $\pm\sqrt{-5}$.

II. **Solution by LEONARD E. DICKSON, M. A.. University of Chicago.**

$$(2)-(1) \text{ gives } w^4+w^2-y^4-y^2=270 \qquad (5)$$
$$(3)-(1) \text{ gives } w^4+w^2-z^4-z^2=182 \qquad (6)$$
$$(4)-(1) \text{ gives } w^4+w^2-x^4-x^2=252 \qquad (7)$$
$$(5)-(6) \text{ gives } z^4+z^2-y^4-y^2=88 \qquad (8)$$
$$(5)-(7) \text{ gives } x^4+x^2-y^4-y^2=18 \qquad (9)$$
$$(1)-(9) \text{ gives } 2y^4+2y^2+z^4+z^2=94 \qquad (10)$$
$$(10)-(8) \text{ gives } 3y^4+3y^2=6 \qquad (11)$$
$$\therefore\ y=\pm1 \text{ or } \pm\sqrt{-2}.$$

From (8) and (11), $z^4+z^2=90$. $\therefore\ z=\pm3$ or $\pm\sqrt{-10}$.
From (9) and (11), $x^4+x^2=20$. $\therefore\ x=\pm2$ or $\pm\sqrt{-5}$.
From (5) and (11), $w^4+w^2=272$. $\therefore\ w=\pm4$ or $\pm\sqrt{-17}$.
Hence there are $4^4=256$ sets of values as solutions.

Also solved by *B. F. Burleson, H. W. Draughon, J. H. Drummond, J. K. Ellwood, M. A. Graber, J. F. W. Scheffer, F. P. Matz*, and *G. B. M. Zerr*.

40. Proposed by B. F BURLESON, Oneida Castle, New York.

Find by quadratics all the possible values for x and y in the equations $x^3+y^3=b=35, \ldots .(1)$, and $x^2+y^2=a=13 \ldots .(2)$.

I. **Solution by the PROPOSER.**

From equation (1) $y=\sqrt[3]{(b-x^3)} \ldots .(3)$. From equation (2) $y=\sqrt{(a-x^2)} \ldots .(4)$. Equating (3) and (4) and clearing from radicals we obtain, $2x^6-3ax^4-2bx^3+3a^2x^2-(a^3-b^2)=0 \ldots .(5)$. Substituting numerical for literal values in (5), it becomes, $2x^6-39x^4-70x^3+507x^2-972=0 \ldots .(6)$. Factoring (6), $(x^2-5x+6)(x^2-2x-4\frac{1}{2})(2x^2+14x+36)=0 \ldots .(7)$. Thus far finding the six roots of equation (6), it is resolved into finding the roots of the three quadratic equations $x^2-5x=-6 \ldots .(8)$, $x^2-2x=4\frac{1}{2} \ldots .(9)$, and $2x^2+14x=-36 \ldots .(10)$. Resolving equation (8) for the two values of x in it, and then substituting these values severally in (3) or (4) for the corresponding values of y, we get, $x=2$ or 3, and $y=3$ or 2. In the same way we find from eq. (9), $x=1+\sqrt{5\frac{1}{2}}=3.345208+$ and $y=1-\sqrt{5\frac{1}{2}}=-1.345208+$, or $x=1-\sqrt{5\frac{1}{2}}=-1.345208+$, and $y=1+\sqrt{5\frac{1}{2}}=3.345208+$. From eq. (10) we obtain the imaginary roots, $x=-3\frac{1}{2}+\frac{1}{2}\sqrt{-23}$, and $y=-3\frac{1}{2}-\frac{1}{2}\sqrt{-23}$, $x=-3\frac{1}{2}-\frac{1}{2}\sqrt{-23}$ and $y=-3\frac{1}{2}+\frac{1}{2}\sqrt{-23}$. Thus x and y have six values each and no more, all of which we have found by quadratics.

II. **Solution by J. K. ELLWOOD, A. M., Colfax-School, Pittsburg, Pennsylvania, and J. W. WATSON, Middle Creek. Ohio.**

Let $x+y=p$, $xy=s$. Then the equations become, $ap-sp=b$, $a+2s=p^2$. Eliminating s, $p^3-3ap+2b=0$, or $p^3-39p+70=0$. It seems the literal solution can not be completed by using quadratics. But multiplying p^3-39p

$+70=0$ by p, $p^4-39s^2=-70p$. Adding to both sides $25p^2+49$, we have $p^4-14p^2+49=25p^2-70p+49$, whence $p^2-7=\pm(5p-7)$ and $p=2$, 5, or -7; hence $s=6$, $-4\frac{1}{2}$, or 18. [Or, from $p^3-39p+70=0$, we have $(p-5)(p^2+5p-14)=0$. $\therefore$ $p=5$, 2, or -7, $s=6, -4\frac{1}{2}$, or 18.]

$\therefore$ $x+y=5$, $xy=6$, $\therefore$ $x=3$ or 2, $y=2$ or 3.

$x+y=2$, $xy=-4\frac{1}{2}$, $\therefore$ $x=\frac{1}{2}(2\pm\sqrt{22})$, $y=\frac{1}{2}(2\pm\sqrt{22})$.

$x+y=-7$, $xy=18$, $\therefore$ $x=\frac{1}{2}(-7\pm\sqrt{-23}$, $y=\frac{1}{2}(-7\pm\sqrt{-23})$.

$\therefore$ There are six values for x and six values for y admissible.

III. Comment by H. W. DRAUGHON, Olio, Mississippi.

The problem can not be solved by quadratics as may be shown thus: The resulting literal equation $p^3-3ap+2b=0$ can not be solved by quadratics, and therefore the given equations can not be solved by quadratics. Cubics of this class can be *apparently* solved by quadratics, when they have one commensurable root. Let r be one root of the equation $p^3=3ap-2b$, for instance. Subtracting r^2p from both members gives $p^3-r^2p=(3a-r^2)p-2b$. Obviously both members of this equation can be divided exactly by $p-r$, giving a quadratic equation, but before this subtraction can be made we must find r, which can not be done by quadratics. If we substitute the definite values for a and b we readily complete the solution.

Also solved by *A. H. Bell*, *P. S. Berg*, *D. G. Durrance, Jr.*, *H. W. Draughon*, *F. P. Matz*, *G. B. M. Zerr*, *J. F. W. Scheffer*, *C. D. Schmitt*, and *H. C. Wilkes*.

PROBLEMS.

50. Proposed by LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago.

Given $b=a\sqrt{-1}.\tan\frac{m\pi}{n}$, m being an arbitrary integer, find the simplest *real* relation between a and b.

51. Proposed by J. W. NICHOLSON, LL. D., President and Professor of Mathematics, Louisiana State University and A. and M. College, Baton Rouge, Louisiana.

Solve the equation $x^5+5mx^3+5m^2x+n=0$.

GEOMETRY.

Conducted by B.F.FINKEL, Kidder, Missouri. All Contributions to this department should be sent to him.

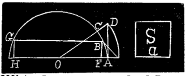
SOLUTIONS OF PROBLEMS.

37. Proposed by B. F. BURLESON, Oneida Castle, New York.

Inscribe in a semi-circle (1). a rectangle having a given area; (2), a rectangle having the maximum area.

Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, and S. N. COLLIER, University of Mississippi.

(1). Let ABC be the given semi-circle with center O; and let S be a square of given area with side a.



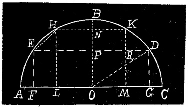
To inscribe in ABC a rectangle equivalent to S.

Construction:—At A, the left-hand extremity of the semi-circumference, draw a tangent AD making it equal to a. Draw OD. With O as center and OD as radius describe an arc cutting OA produced at E. Through E draw a line making an angle of 45° with EO, intersecting the circumference at B. Through B draw a parallel to AO cutting the circumference at G. Through B and G draw perpendiculars to AO meeting the bounding diameter at F and H respectively. Then $FBGH$ is the rectangle required.

Proof:—$OD=\sqrt{(AD^2+OR^2)} = \sqrt{(a^2+R^2)}$, denoting OA by R. But $OD=OE=OF+FE=OF+FB$. $\therefore$ $\sqrt{(a^2+R^2)}=OF+FB$.

Squaring, $a^2+R^2=OF^2+2.OE.FB+FB^2$. But $OF^2+FB^2=R^2$. $\therefore$ $a^2=2.OF.FB$. $2.OF$ is the base of the rectangle and FB is its altitude. Also, a^2=given area. $\therefore$ the rectangle is equivalent to the given area.

(2). Let ABC be a given semi-circle with center O. To inscribe in ABC a maximum rectangle.



Construction:—Draw the radius OD making an angle of 45° with OC. With D as one vertex construct the rectangle $DEFG$. $DEFG$ is the rectangle required.

Proof:—Let $LMKH$ be any other inscribed rectangle; OB the radius perpendicular to OC.

Compare rectangles PG and NM, the halves of the rectangles EG and HM. Rectangle PM is common to the two. Rect. $RG>$ rect. NR; for $DG(=DP)$ $>RP$, and $DR>KR$.

Since $\angle RKD > \angle RDK$, the former being measured by one-half of an arc greater than 90° and the latter by one-half an arc less than 90°.

It follows that $EDGF$ is the rectangle required.

A. L. Foote furnished a neat algebraic solution; *G. B. M. Zerr*, *P. S. Berg*, *Cooper D. Schmitt* solved the problem by the calculus, and *C. D. M. Showalter* gave a good geometrical solution. Space forbids further consideration of this problem.

38. Proposed by LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago.

Give a *strictly geometric* proof of my fundamental theorem of the Inscription of Regular Polygons, viz: Suppose a circle of unit radius divided at the points $A, A_1, A_2, A_3, \ldots . A_p, \ldots$ into $2p+1$ equal parts and the diameter AO drawn. Then, if the chords $OA_1, OA_2, \ldots . OA_p$ be drawn, we have $OA_1 - OA_2 + OA_3 - OA_4 + OA_5 - \ldots . \pm OA_p = 1$.

Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

For plainness we will solve this rpoblem in full for the 13-gon. A general solution is as easy, but not as clearly understood.

Let OA_6, OA_5, OA_4, &c. $= a_1, a_2, a_3$, &c.

$$A_4A_5 = A_5A_6 = A_6A_7 = \&c = c, \quad A_4A_6 = A_5A_7 = A_6A_8 = \&c. = d.$$

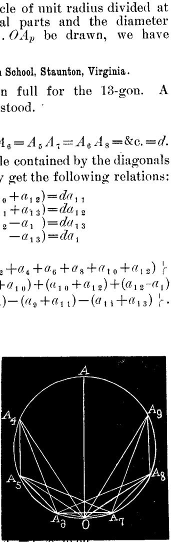
Now by Ptolemy's Theorem:—The rectangle contained by the diagonals of a quadrilateral inscribed in a circle &c., we easily get the following relations:

$$\begin{array}{l|l|l} c(a_1+a_3)=da_2 & c(a_6+a_8\ )=da_7 & c(a_{10}+a_{12})=da_{11} \\ c(a_2+a_4)=da_3 & c(a_7+a_9\ )=da_8 & c(a_{11}+a_{13})=da_{12} \\ c(a_3+a_5)=da_4 & c(a_8+a_{10})=da_9 & c(a_{12}-a_1\ )=da_{13} \\ c(a_4+a_6)=da_5 & c(a_9+a_{11})=da_{10} & c(a_2\ -a_{13})=da_1 \\ c(a_5+a_7)=da_6 & & \end{array}$$

Hence $d\{(a_1+a_3+a_5+a_7+a_9+a_{11}+a_{13})-(a_2+a_4+a_6+a_8+a_{10}+a_{12})\}$
$=c\{(a_2-a_{13})+(a_2+a_4)+(a_4+a_6)+(a_6+a_8)+(a_8+a_{10})+(a_{10}+a_{12})+(a_{12}-a_1)$
$-(a_1+a_3)-(a_3+a_5)-(a_5+a_7)-(a_7+a_9)-(a_9+a_{11})-(a_{11}+a_{13})\}$.

$\therefore\ (d+2c)\{(a_2+a_4+a_6+a_8+a_{10}+a_{12})$
$-(a_1+a_3+a_5+a_7+a_9+a_{11}+a_{13})\}=0$.

$\therefore\ a_2+a_4+a_6+a_8+a_{10}+a_{12}$
$=a_1+a_3+a_5+a_7+a_9+a_{11}+a_{13}$.



Generally $a_2+a_4+a_6+\ldots . +a_{2p}$
$=a_1+a_3+a_5+\ldots . +a_{2p+1}$.

In the above, O can be any point between OA_p and OA_{p+1}.

In the problem $OA=a_7=2$, $a_6=a_8$, $a_4=a_6$, $a_2=a_4$, $a_1=a_{13}$, $a_3=a_{11}$, $a_5=a_9$.

$\therefore\ a_2+a_4+a_6=a_1+a_3+a_5+1$, but $a_6=OA_1$, $a_5=OA_2$, $a_4=OA_3$, $a_3=OA_4$,
$a_2=OA_5$, $a_1=OA_6$.

$\therefore\ OA_1-OA_2+OA_3-OA_4+OA_5$
$-OA_6=1$. p, even.

For 15-gon. $OA_1-OA_2+OA_3-OA_4+OA_5-OA_6+OA_7=1$, p, odd.

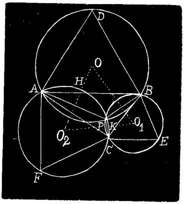
$\therefore\ OA_1-OA_2+OA_3-OA_4+OA_5-\ldots . \pm OA_p=1$, as p is odd or even.

39. Proposed by J. K. ELLWOOD, Principal of Colfax Schools, Pittsburg, Pennsylvania.

If on the three sides of any plane triangle equilateral triangles be described, the lines joining the centres of these equilateral triangles form an equilateral triangle.

I. Solution by F. E. MILLER, Ph. D., Professor of Mathematics, Otterbein University, Westerville, Ohio; Professor J. F. W. SCHEFFER, A. M., Hagerstown, Maryland; JOHN T. FAIRCHILD, Ada, Ohio; J. C. CORBIN, Pine Bluff, Arkansas; and the PROPOSER.

Let ABC be any plane $\triangle$, O, O_1, and O_2 the centres of the equilateral $\triangle$'s constructed. About these $\triangle$'s pass circumferences. They will intersect in a point, P.



Let P be the intersection of the 2 circles, AFC and CEB. Join AP, CP and BP. Since $APCF$ is inscribed, $\angle F + \angle APC = 180^\circ$. But $\angle F = 60^\circ$. $\therefore$ $\angle APC = 120^\circ$. Similarly, $\angle CPB = 120^\circ$. $\therefore$ $\angle APB = 120^\circ$; and $\angle APB + \angle D = 180^\circ$. $\therefore$ $APBD$ is inscribed, and P is in the circumference of DAB. (Q. E. D.)

Lines that join the centres of intersecting circles bisect the common chords and the intercepted arcs. $\therefore$ arc $HP = \frac{1}{2}$ arc AP; and arc $PK = \frac{1}{2}$ arc PC. $\therefore$ arc $HPK = \frac{1}{2}$ arc APC. But arc APC measures the angle $F = 60^\circ$ at circumference; therefore its half HK measures an equal angle at the centre.

$\therefore$ $\angle O_2 = \angle F = 60^\circ$. Similarly, $\angle O_1$ may be shown $= \angle E$, and $\angle O = \angle D$. But equiangular $\triangle$'s are also equilateral. $\therefore$ OO_1O_2 is equilateral.

Q. E. D.

II. Solution by Professor G. B. M ZERR, A. M, Principal of High School, Staunton, Virginia, and Professor J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

Let O, O_1, O_2 (Fig. in Solution I.) be the centres of the equilateral triangles on the sides BA, BC, AC respectively, of the triangle ABC. Let $\triangle =$ area ABC.

Then $\angle OAB = \angle OBA = \angle O_1BC = \angle O_1CB = \angle O_2CA = \angle O_2AC = 30^\circ$.

$$OB = OA = \frac{c}{\sqrt{3}}, \quad O_1B = O_1C = \frac{b}{\sqrt{3}}, \quad O_2C = O_2A = \frac{a}{\sqrt{3}}.$$

$$\therefore \quad OO_1 = \sqrt{\frac{c^2}{3} + \frac{b^2}{3} - \frac{2bc}{3}\cos.(60^\circ + A)}$$

$$= \sqrt{\frac{c^2}{3} + \frac{b^2}{3} - \frac{1}{3}\,bc\cos.\,A + \frac{\sqrt{3}}{3}\,bc\sin A}$$

$$= \sqrt{\frac{1}{6}(a^2 + b^2 + c^2 + \sqrt{3}\triangle)} = O_1O_2 = O_2O$$

$\therefore$ OO_1O_2 is equilateral.

Excellent solutions of this problem were also received from *Professors F. P. Matz* and *G. I. Hopkins.*

PROBLEMS.

44 Proposed by I. J. SCHWATT, Ph. D., Professor of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania.

(1). If from the middle point M of the side BC of the triangle ABC a parallel to the bisector AF of the external angle to ABC is drawn to meet AB at K, the point K divides then the side AB in KA

$$=\tfrac{1}{2}(AB+AC) \text{ and } KB=\tfrac{1}{2}(AB-AC).$$

(2). If K is joined to the extremity D of the diameter perpendicular to BC then is KD perpendicular to AB.

45. Proposed by B. F. BURLESON, Oneida Castle, New York.

Determine the radius of a circle circumscribing three tangent circles of radii $a=15$, $b=17$, and $c=19$.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

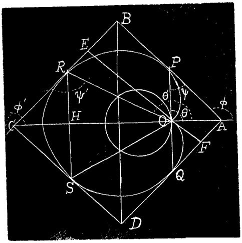
SOLUTIONS OF PROBLEMS.

30. Proposed by E. W. NICHOLS, Professor of Mathematics in the Virginia Military Institute, Lexington, Virginia.

Given the cardioid $r=a(1-\cos\theta)$; find the area of its circumscribing square formed by tangents making angles of $45°$ with its axis.

I. Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let $OPRSQ$ be the cardioid. Draw PQ through the cusp perpendicular to the initial line AC. From the property of the cardioid the angle APO, made by the tangent and radius vector at P, $=\frac{1}{2}\angle POA$. But $\angle AOP=\frac{1}{2}\pi$. $\therefore$ $\angle OPA=\angle OAP=\frac{1}{4}\pi$. $\therefore$ the tangents BA, DA at the points P, Q are inclined at an angle of $45°$ to the axis and are perpendicular to each other. Draw the radii vectors OR, OS, making the $\angle ROP=\angle SOQ=\frac{1}{3}\pi$, and draw the tangents CB, CD at the points R, S. Then $\angle ROP=\angle SOQ=\frac{1}{3}\pi$, $\angle OPB=\angle OQD=\frac{3}{4}\pi$, $\angle ORB=\angle OSD=\frac{5}{12}\pi$.



$$\therefore \angle ROP+\angle OPB+\angle ORB=\angle SOQ+\angle OQD+\angle OSD=\tfrac{3}{2}\pi.$$

$\therefore$ $\angle B=\angle D=\frac{1}{2}\pi$, an $ABCD$ is the required square.

Now $OR=a(1-\cos\frac{5}{6}\pi)=a(1+\frac{1}{2}\sqrt{3})$, $OQ=a(1-\cos\frac{3}{2}\pi)=a$.

$$OE=OR\sin ORB=OR\sin 75^\circ=\frac{a(2+\sqrt{3})}{2}+\frac{1+\sqrt{3}}{2\sqrt{2}}=\frac{5+3\sqrt{3}}{4\sqrt{2}}a.$$

$$OF=OQ\sin OQF=OQ\sin 45^\circ=\frac{a}{\sqrt{2}}.$$

$$AB=OE+OF=\frac{3(3+\sqrt{3})}{4\sqrt{2}}a.$$

Area square $=AB^2=\dfrac{27(2+\sqrt{3})}{16}a^2$. A solution without the use of Calculus.

II. **Solution by Cadet A. R. GATEWOOD, Virginia Military Institute, Lexington, Virginia; and COOPER D. SCHMITT, A. M., University of Tennessee, Knoxville, Tennessee.**

$$\operatorname{Tan}\Psi=r\Big/\frac{dr}{d\theta}\cdot\frac{dr}{d\theta}=a\sin\theta.\quad\therefore\ \tan\Psi=\frac{r}{a\sin\theta}=\frac{2a\sin^2\frac{\theta}{2}}{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$=\tan\frac{\theta}{2}.\quad\therefore\ \Psi=\frac{\theta}{2}.\quad \phi=\Psi+\theta=\frac{\theta}{2}+\theta=\frac{3\theta}{2}.$$

Now when tangents make angles of 45° each with the initial line, $\phi=135^\circ$, $\phi'=225^\circ$. $\therefore\ \theta=90^\circ$, $\theta'=150^\circ$, $\angle ROH=30^\circ$.

When $\theta=90^\circ$, $OA=OP=a$. When $\theta=150^\circ$, $RO=a(1-\cos150^\circ)=a\left(1+\frac{\sqrt{3}}{2}\right)$,

$$CH=RH=RO\sin30^\circ=\tfrac{1}{2}a\left(1+\frac{\sqrt{3}}{2}\right),\ HO=RO\cos30^\circ=\frac{\sqrt{3}}{2}a\left(1+\frac{\sqrt{3}}{2}\right).$$

Now $CA=CH+HO+OA=\frac{1}{2}a\left(1+\frac{\sqrt{3}}{2}\right)+\frac{\sqrt{3}}{2}a\left(1+\frac{\sqrt{3}}{2}\right)+a=\frac{3}{4}a(3+\sqrt{3})$.

$ABCD=BA^2=\frac{1}{2}CA^2=\frac{1}{2}\left[\frac{3}{4}a(3+\sqrt{3})\right]^2=\frac{27}{16}(2+\sqrt{3})a^2$, which is the area of the circumscribed square.

III. **Solution by F. P. MATZ, Ph. D., New Windsor College, New Windsor, Maryland; ALFRED HUME, C. E., University of Mississippi, P. O., Mississippi; and J. SCHEFFER, A. M., Hagerstown, Maryland.**

Since the sides of the circumscribing square are to intersect the axis at angles of 45°, we have from *Todhunter's Differential Calculus*, p. 304, Art. 278, that

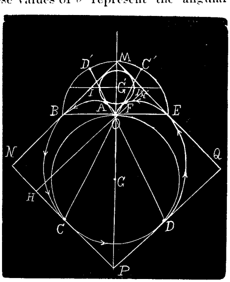
$$\tan\Psi=\frac{dy}{dx}=\frac{\sin\theta(dr/d\theta)+r\cos\theta}{\cos\theta(dr/d\theta)-r\sin\theta},$$

$$=\frac{\sin\theta(a\sin\theta)+a(1-\cos\theta)\cos\theta}{\cos\theta(a\sin\theta)-a(1-\cos\theta)\sin\theta}=\pm1\ldots(1).$$

From (1) we have, respectively:

$\tan\frac{3}{2}\theta=+1$. $\therefore\ \theta=\frac{1}{6}\pi,\ \frac{5}{6}\pi,\ \frac{9}{6}\pi$, etc., and $\tan\frac{3}{2}\theta=-1$. $\therefore\ \theta=\frac{3}{6}\pi,\ \frac{7}{6}\pi,\ \frac{11}{6}\pi$, etc.

Taken in the order of their magnitude, these values of θ represent the angular position of the points A, B, C, D, E, and F, with respect to the origin of polar co-ordinates and the axis of the cardioid. If any tangents to the cardioid be drawn through *these* points, such tangents make angles of 45° with the axis MP. When the origin of polar co-ordinates is at O, the radius vectors of the points already specified becomes respectively:



$OA=\frac{1}{2}(2-\sqrt{3})a$, $OB=a$,

$OC=\frac{1}{2}(2+\sqrt{3})a$, $OD=\frac{1}{2}(2+\sqrt{3})a$,

$OE=a$, and $OF=\frac{1}{2}(2-\sqrt{3})a$.

Consequently the area of the required circumscribing square

$=(MN)^2=(OH+OH')^2$

$=\frac{9}{32}(3+\sqrt{3})^2a,=\frac{27}{16}(2+\sqrt{3})a^2\ldots.(2)$;

and of this square, the diagonal $MP=\frac{3}{4}(3+\sqrt{3})a$. Since the *diagonal* $II'=\frac{3}{4}(3+\sqrt{3})a$, the area of the square the center of the inscribed circle of which is at $G'=(MI)^2=\frac{9}{32}(3-\sqrt{3})^2a^2$, $=\frac{27}{16}(2-\sqrt{3})a^2\ldots.(2)$.

Represent the area of the *larger* square by A and that of the *smaller* square by A'; then from the results given, $A:A'::(2+\sqrt{3}):(2-\sqrt{3})::OC:OF$.

IV. Second Solution by Professors F. P. MATZ; and C. E. WHITE, Trafalgar, Indiana.

The *pedal equation* of the cardioid in consideration, is $p^2=r^3/2a$; that is, for the points B, C, D, and E, we have respectively: $p_B=a/\sqrt{2}$, $p_C=\frac{1}{4}a\sqrt{(26+15\sqrt{3})}$, $p_D=\frac{1}{4}a\sqrt{(26+15\sqrt{3})}$, and $p_E=a/\sqrt{2}$. In order that the quadrilateral circumscribing the cardioid may be a square, we must have $p_B+p_D=p_C+p_E$; and this condition is fulfilled. Hence the *required* area becomes $A=(p_B+p_D)^2=(p_C+p_E)^2=\frac{27}{16}(2+\sqrt{3})a^2\ldots.(1)$,

while the area of the *smaller* square becomes

$$A'=(p_B-p_F)^2=(p_E-p_A)^2=\frac{27}{16}(2-\sqrt{3})a^2\ldots.(2).$$

NOTE—By using the equation $p=2a\sin^3\phi$, in which $\phi=\frac{1}{2}\theta$, a *third* solution can be made. Sufficient data are given in the problem to enable us to make a *fourth* solution, without having recourse to the differential calculus.

Also solved by *O. W. Anthony*, *H. W. Draughon*, and *J. B. Faught*. We regret that *Professor Faught's* solution was mislaid and could not be considered in selecting papers for publication.

PROBLEMS.

41. Proposed by F. P. MATZ, M. Sc., Ph.D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

The *closed portion* of the curve known as "The Cocked Hat," equation

$$x^4+x^2y^2+4ax^2y-2a^2x^2+3a^2y^2-4a^3y+a^4=0,$$
revolves around the axis of y. Find the *companulate* volume generated. If the same portion of the curve revolve around the axis of x, find the *fusiform* volume generated. Also, determine the area of this closed portion of the curve.

41. Proposed by F. M. SHIELDS, Coopwood, Mississippi.

A railroad turn-table 100 feet long is balanced upon a pivot in the center of a circular track 100 feet in diameter. How far does a man walk who starts at one end of the table and walks, at a uniform rate, the entire length of the table in the same time that the table makes two revolutions, if the table starts to turn at the same time the man starts to walk?

MECHANICS.

Conducted by B.F.FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

16. Proposed by A. H. BELL, Hillsboro, Illinois.

An iron bar 20 feet long and weighing 2,000 lbs, leans against a wall at angles of 30°, 45°, and 80°. Determine the *pressure* upon the floor, and *that* upon the wall.

I. Solution by F. P. MATZ, M. Sc., Ph.D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Let P_F=the pressure upon the floor, and P_W=the pressure upon the wall, then, if θ be the angle the bar makes with ths vertical wall, we have from *Wood's Analytical Mechanics*, the equations:

$P_F=-W=-2000$ lbs; that is, the floor sustains the whole weight of the bar, and $P_W=\frac{1}{2}W\tan\theta$....(1).

According to the conditions of the problem, we deduce from (1) the following results:

For θ = 1°, we have P_W= 17.455054+lbs;
" " =10°, " " " = 176.327696+ " ;
" " =30°, " " " = 577.350266+ " ;
" " =45°, " " " =1000.000000+ " ;
" " =60°, " " " =1732.050800+ " ;
" " =80°, " " " =5671;280256+ " ;
" " =89°, " " " =5728.996052+ " ;

NOTE.—How the *last two* results, and all *succeeding* results to the limit of the quadrant, are to be interpreted, is a question on which Professor DeVols in Wood can *interest* the readers of the MONTHLY.

II. Solution by EDMOND FISH, Hillsboro, Illinois.

It must be assumed that the bar is prevented from sliding down by some lateral resistance, either the roughness of the floor, or some object, as a

rod or plank laid between the foot of the bar and an opposite wall. Now let a rope be passed from the foot of the bar over a pulley. What weight attached to the rope will relieve the floor of all pressure?

By the Law of Virtual Velocities, Power and Weight are inversely proportioned to their rates of movement in the direction required. In the present case these rates are obviously equal, and the power must equal the weight. Hence the pressure on the floor will be 2000 lbs. for all positions of the bar.

For the pressure on the wall, suppose the bar inclined 30° from the vertical. Its whole weight may be considered concentrated in its center of gravity. Its tendency to move is in an arc whose radius is 10. But only the vertical part of this movement is effective, and this part is expressed by sin30°.

This supposed movement of the center of the bar implies a movement of the upper end in an arc whose radius is 20. But we are concerned only with the horizontal part of this movement and this is expressed by cos30°. Hence, by the law before quoted, Power (2000 lbs): Weight :: 2cos 30° : sin 30°. (*Twice* cos 30° because R is double).

$$\therefore \text{ Weight or pressure on wall} = \frac{2000 \times \sin 30^\circ}{2 \cos 30^\circ} = 1000 \times \tan 30^\circ.$$

In general the pressure on the wall equals half the weight of the bar multiplied by the tangent of the inclination.

At 30° A is $577\frac{35}{100}$ lbs.
At 45° A is 1000 lbs.
At 80° A is $5671\frac{28}{100}$ lbs.

This problem was also solved by *P. S. Berg and J. F. W. Scheffer.*

17. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Find the law of density of strings collected into a heap at the edge of a table with the end of the string just over the edge, so that equal masses may always pass over in equal units of times.

Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

A pulverized solid, if piled up, will settle by the force of gravity to a certain inclination, according to the smallness and smoothness of its particles. Now, a string is practically a cylindric solid of *great* length, *small* cross-sections, and *indefinite* flexibility and compresibility. In passing over the edge of the table, the string will pile into an *approximate* right circular cone. A stream of pulverized solid—an *impalpable powder*, like precipitated B_aSO_4—is practically a *string* of indefinitely small molecular attraction; and such a string will pile into a right circular cone.

Let $AO=r_1$, $CI=m_1$, $\angle CAO=\omega_1$ and δ_0=the initial density of the first *string-cone* formed. Pass a pound (W_1) of string *uniformly* over the edge of the table; then from the cone $AOB-C$, we have

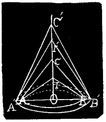
$$W_1=M_1g=V_1\delta_0,=\tfrac{1}{3}\pi r_1^3\delta_0 g\tan\omega_1\;\ldots\ldots(1).$$

Pass similarly a second pound of string; then will be formed the cone

$A'OB'-C'$, and the cone $AOB-C$ will be compressed into the cone $AOB-I$.

The volume of the *compressed* string-cone, $AOB-I$, becomes

$W_1=M_1g=V_1\delta_1 g_1=\frac{1}{3}\pi r_1^2(r_1\tan\omega_1-m_1)\delta_1 g\ldots\ldots(2)$.



Equating the right-hand members of (1) and (2), we have

$\delta_1=\left(\dfrac{r_1\tan\omega_1}{r_1\tan\omega_1-m_1}\right)\delta_0,\ldots\ldots,\delta_n=\left(\dfrac{r_1\tan\omega_1}{r_1\tan\omega_1-m_n}\right)\delta_1\ldots\ldots(3)$.

The values of $\delta_1\ldots\ldots\delta_n$, as determined empirically for determinate conditions, immediately lead to the required law of density.

18. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Mississippi.

An elleptic paraboloid whose equations is $\dfrac{y^2}{a}+\dfrac{x^2}{b}=2z$ has its axis vertical and vertex downward. If μ be the co-efficient of friction, prove that a heavy particle will rest at any point of the surface below its intersection with the cylinder $\dfrac{y^2}{a^2}+\dfrac{x^2}{b^2}=\mu^2$.

Solution by the PROPOSER.

If W is the weight of the particle, N and T its normal and tangential components, $W^2=N^2+T^2$. Also, when the particle is on the point of sliding, $T=\mu N$. Hence, $W^2=(1+\mu^2)N^2$. Again, $W\cos\theta=N$, θ being the angle between the normal and the Z axis.

Now $\cos\theta=\dfrac{\dfrac{dF}{dz}}{\sqrt{\left(\dfrac{dF}{dx}\right)^2+\left(\dfrac{dF}{dy}\right)^2+\left(\dfrac{dF}{dz}\right)^2}}$, $F(x,y,z)=0$ being the equation of the surface, and the differential-coefficients being partial.

$\dfrac{dF}{dx}=\dfrac{2x}{b}$, $\dfrac{dF}{dy}=\dfrac{2y}{a}$, $\dfrac{dF}{dz}=-2$. Substituting, $\cos\theta=\dfrac{-2}{\sqrt{\left(\dfrac{4x^2}{b^2}+\dfrac{4y^2}{a^2}+4\right)}}$.

This, in the fourth equation above, gives, after squaring,

$N^2=\dfrac{4W^2}{\dfrac{4x^2}{b^2}+\dfrac{4y^2}{a^2}+4}$. Substituting this value of N^2 in the third equation and reducing we get $\dfrac{y^2}{a^2}+\dfrac{x^2}{b^2}=\mu^2$.

This is the relation between the x and y co-ordinates of every point of the surface at which the friction is limiting; in other words, these points lie on the cylindrical surface of which this is the equation. Consequently, their locus

is the curve of intersection of the paraboloid and the cylinder. This curve divides the given surface into two parts. The particle will be in equilibrium at any point of the lower and at no point of the upper.

Two excellent solutions of this problem were received from *F. P. Matz*, and one from *G. B. M. Zerr*.

PROBLEMS.

26. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

If an elastic sphere be electrified in such a manner that the initial internal pressure remains constant, determine an expression for the *ratio of the electrical densities* when the volume of the sphere has been increased to $(m+1)$ times its initial volume.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

20. Proposed by G. B. M. ZERR, A. M., Principal of High School, Staunton. Virginia.

Find two integral numbers, whose sum, difference, and difference of their squares shall be a square, cube, and fourth power.

I. Solution by J. H. DRUMMOND; H. C. WILKES; and M. A. GRUBER.

Let x and y=the two integral numbers. Any number to be a square, a cube, and a fourth power, must also be a twelfth power.

$$\text{Then } x+y=a^{12}$$
$$x-y=b^{12}$$
$$x^2-y^2=a^{12}b^{12}=(ab)^{12}.$$
$$\text{Whence } x=\tfrac{1}{2}(a^{12}+b^{12}),\text{ and } y=\tfrac{1}{2}(a^{12}-b^{12}).$$

In order that x and y be integral, a^{12} and b^{12} must be both odd or both even.

Put $a^{12}=5^{12}=244,140,625$ and $b^{12}=3^{12}=531441$. Then $x=122,336,033$ and $y=121,804,592$. Put $a^{12}=6^{12}=2,176,782,336$ and $b^{12}=2^{12}=4096$. Then $x=1,088,393,216$ and $y=1,088,389,120$.

The lowest values of x and y are found by putting

$$x+y=a^{12}=3^{12}=531441,$$
$$x-y=b^{12}=1^{12}=1.$$
$$\text{Whence } x=265721 \text{ and } y=265720.$$

Many answers can be obtained but the work will be tedious.

II. **Solution by the PROPOSER.**

Let, $x^m+\frac{m(m-1)}{2}x^{m-2}y^2+\ldots+\frac{m(m-1)}{2}x^2y^{m-2}+y^m$, and $mx^{m-1}y+$

$\frac{m(m-1)(m-2)}{2.3}x^{m-3}y^3+\ldots+\frac{m(m-1)(m-2)}{2.3}x^3y^{m-3}+mxy^{m-1}$ be the numbers.

Then $(x+y)^m$, $(x-y)^m$, $(x^2-y^2)^m$, is their sum, their difference, and the difference of their squares. m must be divisible by 2, 3, and 4; this is the case when $m=12$. Then the numbers are $x^{12}+66x^{10}y^2+\ldots+66x^2y^{10}+y^{12}$,and $12x^{11}y+220x^9y^3+\ldots+220x^3y^9+12xy^{11}$ and $(x+y)^{12}$, $(x-y)^{12}$, $(x^2-y^2)^{12}$ is their sum, their difference, and the difference of their squares.

Let $x=2$, $y=1$, then the numbers are 265721, 265720. Their sum $=(3)^{12}=531441=(729)^2=(81)^3=(27)^4$. Their difference $=1=(1)^2=(1)^3=(1)^4$. Difference of their squares $=531441=(729)^2=(81)^3=(27)^4$.

Many other numbers can be found satisfying the conditions.

Also solved by *H. W. Draughon*, and *J. F. W. Scheffer*.

21. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

Find (1) nine positive *integral numbers* in arithmetical progression the sum of whose squares is a *square number*; and (2) find nine integral *square numbers* whose sum is a *square number*.

I. **Solution by H. W. DRAUGHON, Olio, Mississippi.**

(1). Let, $x+4y$, $x+3y$, $x+2y$, $x+y$, x, $x-y$, $x-2y$, $x-3y$, and $x-4y$ be the numbers. Then we are to make the sum of their squares, $9x^2+60y^2=\square$. Let us assume, $9x^2+60y^2=(3x+6m)^2=9x^2+36mx+36m^2$; then,

$x=\frac{60y^2-36m^2}{36m}=\frac{5y^2}{3m}-m$. In order that x may be integral put, $y=3pm\ldots(1)$.

Then, $x=15p^2m-m=(15p^2-1)m\ldots.(2)$. p and m can have any positive, integral values that will make $x>4y$.

Let us make, $p=1$ and $m=1$; then, $x=14$, $y=3$, and the numbers are 2, 5, , 11, 14, 17, 20, 23, and 26. The sum of their squares is $(48)^2$.

Again, let, $p=2$, and $m=1$; then, $x=59$, $y=12$, and the numbers are, 11, 23, 35, 47, 59, 71, 83, 95, and 107. The sum of the squares of this set is $(183)^2$. An infinite number of sets can be thus obtained from (2).

(2). In the *Mathematical Messenger*, Vol. 7, No. 5. page 47, I find the following formula for n square numbers whose sum is a square:

$S+\frac{1}{4}(p-q)^2=\frac{1}{4}(p+q)^2$, in which $S=pq=$the sum of $n-1$ square numbers. Here, $n=9$. Let us assume, $S=(3)^2+(4)^2+(5)^2+(8)^2+(9)^2+(10)^2+(11)^2+(12)^2=560=10\times56$. Let us make $p=56$ and $q=10$; then, we have, $S+(23)^2=(33)^2$. Any other factors of 560 may be taken. When factors give fractional results we clear of fractions.

II. **Solution by R. J. ADCOCK, Larchland, Illinois.**

$a, a+x, a+2x, a+3x, \ldots a+(n-1)x$, is an arithmetical progression of n terms, their sum of squares is $a^2+(a+x)^2+(a+2x)^2+(a+3x)^2+\&c.\ldots$ $+[a+(n-1)x]^2=S=na^2+(n^2-n)ax+\frac{1}{6}(2n^3-3n^2+n)x^2$ by method of difference. This sum is made=a rational square by the method given in Encyclopedia Britannica, Vol. I, Algebra article 121, 9th edition. When n is a square number $na^2+(n^2-n)ax+\frac{1}{6}(2n^3-3n^2+n)x^2=(n^{\frac{1}{2}}a+hx)^2$

$=na^2+2ahn^{\frac{1}{2}}x+\frac{1}{6}(2n^3-3n^2+n)x^2$. Then $x=\dfrac{2ahn^{\frac{1}{2}}-(n^2-n)a}{\frac{1}{6}(2n^3-3n^2+n)-h^2}=\frac{3}{2}$ for $a=1$, $n=9$, $h=14$. The algebraic value of x in expressions for S_1 after clearing denominatives gives a general solution, the numerical $\frac{3}{2}$, gives
$2^2+5^2+8^2+11^2+14^2+17^2+20^2+23^2+26^2=48^2$.

Also solved by *F. P. Matz. G. B. M. Zerr, J. H. Drummond, C. D. Schmitt, J. Scheffer, and M. A. Gruber*

PROBLEMS.

30. Proposed by COOPER D. SCHMITT, Knoxville, Tennessee.

A and B are two integers, A consisting of $2m$ figures each being 1, and B consisting of m figures each being 4. Prove that $A+B+1$ is a square.

31. Proposed by M. A. GRUBER, War Department, Washington, D. C.

How many scalene triangles, of integral sides, can be formed with an altitude of 12? How many isosceles triangles?

AVERAGE AND PROBABILITY.

Conducted by B.F.FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

17. Proposed by A. L. FOOTE, No. 80, Broad St. New York.

A person 30 years of age has an annuity for 10 years, the present worth of which is $1000, provided he lives but ten years; for, if he dies, the annuity ceases What is the annuity worth, on the assumption that 75 out of every 4385 persons die annually, between the ages 30 and 40 years?

Solution by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Kidder Institute, Kidder, Missouri.

Let S=the annuity, the present value, P, of which, for $10,=n$, years is $100.

Then $P=\frac{S}{R-1}\left(\frac{R^n-1}{R^n}\right)$, whence $S=\frac{rPR^n}{R^n-1}$, where $R=1+r$. If we assume $r=.0$, we find $S=\$149.0294$ nearly.

The limit of B's life is $4385\div75=58\frac{7}{15}$ years, $=l$. $\therefore$ the probability that B will be living at the end of 1 year is $\frac{l-1}{l}$; at the end of two years, $\frac{l-2}{l}$; at the end of three years, $\frac{l-3}{l}$; etc.

The present worth of S, $=\$149.20207$, due in 1, 2, 3, 4, etc., years is $\frac{S}{R}$, $\frac{S}{R^2}$, $\frac{S}{R^3}$, etc., to $\frac{S}{R^n}$. The present worth of S due at the end of any year multiplied by the probability of B's living to the end of that year is the actual value of S.

$$\therefore\ S'=\frac{S(l-1)}{Rl}+\frac{S(l-2)}{lR^2}+\frac{S(l-3)}{lR^3}+\ldots\ldots\frac{S(l-n)}{lR^n}\ldots\ldots(1).$$

$\frac{S'}{R}=\frac{S(l-1)}{lR^2}+\frac{S(l-2)}{lR^3}+\frac{S(l-3)}{lR^4}\ldots\ldots\frac{S(l-n)}{lR^{n+1}}\ldots\ldots(2)$, by multiplying (1) by $\frac{1}{R}$.

$$\therefore\ S'\left(\frac{R-1}{R}\right)=\frac{S(l-1)}{Rl}-\frac{S}{lR^2}-\frac{S}{lR^3}-\ldots\ldots-\frac{S(l-n)}{lR^{n+1}},$$

$$=\frac{S}{l}\left\{\frac{l}{R}-\frac{1}{R}-\frac{1}{R^2}-\frac{1}{R^3}\ldots\ldots\frac{1}{R^n}-\frac{l-n}{R^{n+1}}\right\},$$

$$=\frac{S}{l}\left\{\frac{l}{R}-\frac{l-n}{R^{n+1}}-\frac{1}{R}-\frac{1}{R^2}\ldots\ldots\frac{1}{R^n}\right\},$$

$$=\frac{S}{l}\left\{\frac{l}{R}-\frac{l-n}{R^{n+1}}-\frac{1}{R}\left[1-\frac{1}{R^n}\right]\div\frac{R-1}{R}\right\},$$

$=\frac{S}{R}\left\{\left(1-\frac{l-n}{lR^n}\right)-R\left(1-\frac{1}{R^n}\right)\div(R-1)l\right\}$, whence $S'=S/r\{[1-(l-n)/lR^n]-R(1-1/R^n)/rl\}$, $=\$911.881029$.

No solutions of this problem were received from our contributors.

18. Proposed by H. W. DRAUGHON, Clinton, Louisiana.

The probability that A will speak the truth is twice the probability that B will, in an independent statement, speak the truth; but, if A exerts his influence, the probability is that B will agree with him in any statement. What is the probability of the truth of their concurrent testimony, the chances being equal that A may or may not be interested in the matter?

Solution by P. H. PHILBRICK, C. E., Lake Charles, Louisiana.

1. Suppose that A is not interested in the matter. Let $x=$the probability of the truth of any one of B's statements. Then $2x=$the probability of the truth of any one of A's statements. The event did occur if both witnesses tell the truth the probability of which is $x\times2x=2x^2$.

The event did not occur if both witnesses testify falsely the probability of which is $(1-x)(1-2x)$. Hence the probability of the occurrence of the event supposing x to be known is, $p'=\frac{2x^2}{2x^2+(1-x)(1-2x)}$.

Now as the veracity of A may vary from 0 to 1 x, may vary from 0 to $\frac{1}{2}$, and, therefore, the required probability is

$$p=\int_0^{\frac{1}{2}}p'dx\div\int_0^{\frac{1}{2}}dx=4\int_0^{\frac{1}{2}}\frac{x^2dx}{2x^2+(1-x)(1-2x)}=64\int_0^{\frac{1}{2}}\frac{x^2dx}{(8x-3)^2+7}.$$

Let $8x-3=y$, then $x=\frac{1}{8}(y+3)$, $dx=\frac{1}{8}dy$ and the limits of y are 1 and -3.

$$\text{Hence, } p=\tfrac{1}{8}\int_{-3}^{+1}\frac{(y+3)^2dy}{y^2+7}=\tfrac{1}{8}\int_{-3}^{+1}\left(1+\frac{6y}{y^2+7}+\frac{2}{y^2+7}\right)dy$$

$$=\left[\tfrac{1}{8}y+\tfrac{3}{8}\log_e(y^2+7)+\frac{1}{4\sqrt{7}}\tan^{-1}\frac{y}{\sqrt{7}}\right]_{-3}^{1}=\tfrac{1}{2}-\tfrac{3}{8}\log_e2+\frac{1}{4\sqrt{7}}\tan^{-1}\sqrt{7}.$$

2. Suppose that A is interested in the matter. In this case B's testimony agrees with that of A, and A's alone is to be considered. The probability of the truth of A's testimoney is $P'=2x$ in which x may vary from 0 to $\frac{1}{2}$. Hence, the required probability is $P=\int_0^{\frac{1}{2}}P'dx\div\int_0^{\frac{1}{2}}dx=\left[2x^2\right]_0^{\frac{1}{2}}=\frac{1}{2}$.

Since the chances are equal that A may be or may not be interested in the matter, the probability required is equal to the half sum of the preceeding results, or $p_1=\frac{1}{2}\left(1-\frac{3}{8}\log_e2+\frac{1}{4\sqrt{7}}\tan^{-1}\sqrt{7}\right)$.

PROBLEMS.

28. Propssed by G. I. HOPKINS, Instructor in Mathematics and Physics in High School, Manchester, N. H.

John and Henry passed in to their teacher a written exercise in geometry. On glancing over the two papers, the teacher noticed a striking resemblance in the hand writing. On closer inspection, he found the two diagrams precisely alike each employing the same nine letters, the points being designated by the same letters in both. The teacher at once suspected that Henry had asked and received assistance from John, but refraines from saying anything until he had examined the papers thoroughly. On further inspection of John's paper he discovered two mistakes: then turning to Henry's paper he discovered the same two mistakes in his. The next morning John denied that he had written that exercise for Henry but acknowledged that he had done so once before. When asked by the teacher, however, to pick out his paper from a number of others, the autographs all being concealed, he picked out the paper to which Henry's name was signed. Henry was absent from school that morning, and has not yet returned. What is the probability that John did *not* write Henry's exercise for him?

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

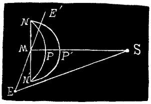
SOLUTIONS OF PROBLEMS.

15. Proposed by SAMUEL HART WRIGHT, M. D., M. A., Ph. D., Penn Yan, Yates County, New York.

Required the illuminated area of the Moon's disc when $\frac{2}{3}$ through its first quarter, or 60°of longitude east of the Sun, the Earth and Moon being at their mean distances.

Solution by F. P. MATZ, M. Sc.; Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Represent the centers of the sun, earth, and moon, by S, E, and M, respectively; also, represent the *mean* distances of the moon and sun from the earth, respectively, by $EM=d=1$, and $SE=nd=395.40$. Let $\angle MES=\Psi=60^\circ$; $\angle SME'=\phi$; and $MS=d\sqrt{(n^2+1-2n\cos\Psi)},=d\sqrt{(n^2-n+1)}=394.90$; then, by well-known principles of Trigonometry, we deduce



$$\phi=\sin^{-1}\left(\frac{n\sin\Psi}{\sqrt{[(n-1)^2+2n(1-\cos\Psi)]}}\right),=60^\circ 7' 33''.223.$$

Put $\mathbf{A},=\frac{1}{2}$, the apparent area of the semi-circle $NP'N'$ at a unit's distance, $\mathbf{A}'$ =the apparent area of this semi-circle at a distance $EM=d$; then, since the apparent *diameter* of the moon varies inversely as the distance and the apparent *area* of the moon's disc varies as the square of the distance, we we have $\mathbf{A}'=\mathbf{A}/d^2$. The *phase*, or illuminated area, $NP'N'PN$ at a distance $EM=d$, we represent by $\mathbf{P}$; and the apparent area of the semi-ellipse NPN' at a distance $EM=d$, by $\mathbf{E}$; then, according to obvious principles and deductions already made, we obtain the formula,

$$\mathbf{P}=\mathbf{A}'-\mathbf{E}=\mathbf{A}'(1-\cos\phi)=\mathbf{A}(1-\cos\phi)/d^2,$$

$$=\frac{\mathbf{A}}{d^2}\left[1-\sqrt{\left(\frac{(n-1)^2-n(1-\cos\Psi)[n(1+\cos\Psi)-2]}{(n-1)^2+2n(1-\cos\Psi)}\right)}\right]$$

$=\mathbf{A}(1-.4981517)/d^2=\frac{1}{2}of.5018483=.2509241+$; that is, about *one-forth of the apparent disc* of the moon is then illuminated.

NOTE.—An easy trigonometrical operation gives the formula,

$$\phi=\pi-\left\{\Psi+\tan^{-1}\left[\left(\frac{n-1}{n+1}\right)\cot\tfrac{1}{2}\Psi\right]\right\}$$

$$=\pi-\left\{\Psi+\tan^{-1}\left[\left(\frac{n-1}{n+1}\right)\sqrt{\left(\frac{1+\cos\Psi}{1-\cos\Psi}\right)}\right]\right\},$$

$$=\pi-[\tfrac{1}{3}\pi+\tan^{-1}(\tfrac{986}{991}\sqrt{3})]=60^\circ 7' 33''.223.$$

It must be observed that a reversed crescent of the same size is illuminated when the "Waning Moon" is two-thirds through her *last* quarter.

PROBLEMS.

28. Proposed by "IAGO"-(The late DR. JAMES MATTESON, DeKalb Center, Illinois.)

If 9 gentlemen, or 15 ladies, will eat 17 apples in 5 hours, and 15 gentlemen and 15 ladies can eat 47 apples of a similar size in 12 hours, the apples growing uniformly; how many boys will eat up 360 apples in 60 hours, admitting that 120 boys can eat the same number as 18 gentlemen and 26 ladies? *F. P. Matz.*

29. Proposed by ALEXANDER MACFARLANE, M. A., D. Sc., LL.D., Cornell University, Ithaca. New York.

A rectangular room has the four walls, the floor, and the ceiling covered with mirrors; a candle is placed inside the room: find a formula which will express all the images.

EDITORIALS.

Professor Milton L. Comstock, Professor of Mathematics, Knox College, Galesburg, Illinois, says, I have read the MONTHLY from the beginning and I am willing to bear testimony as to its excellence.

In future numbers of the MONTHLY, some valuable contributions on important Mathematical subjects may be looked for from Dr. G. A. Miller, of the University of Michigan, and Dr. W. B. Smith of the Tulane University of Louisiana.

Professor F. P. Matz, of the Department of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland, has just received official notice of his unanimous election as member of the London Mathematical Society, having been proposed for membership, by Professor J. J. Sylvester. Dr. Matz was also elected member of the American Mathematical Society at its meeting in February, having been proposed for election by Professor William W. Johnson, of the United States Naval Academy and endorsed by Professor Simon Newcomb. We congratulate Dr. Matz on these merited recognitions.

Professor J. E. Oliver, Professor of Mathematics in Cornell University, died, on the 27th of March, 1895, after an illness of ten weeks.

On Monday, April 8th, 1 95, Dr. Alexander Macfarlane, was married to Miss Helen Martha Swearingen, of San Antoneio, Texas. Dr. Macfarlane and his wife will make their home in Ithaca, New York. The MONTHLY family wishes them a long and happy life.

BOOKS AND PERIODICALS.

A College Algebra. By J. M. Taylor, A. M., Second edition, 1892. 8vo. cloth, 318 pp. Price, $1.50. Boston and Chicago: Allyn & Bacon.

The first part of this Algebra embraces an outline of those fundamental principles of the science that are usually required for admission to a college or scientific school. The subject of Equivalent Equations and Equivalent Systems of Equations are fully treated. In the Second Part, the author has given a full discussion of the Theory of Limits which is followed by one of its most important appplications, Differentiation. Pages 266-317 are given to the Theory of Equations. The subjects treated in this book are presented with the utmost clearness and simplicity. Teachers of Higher algebra should examine this work. It is neatly printed and well bound in substantial cloth. B.F.F.

Theorems in the Calculus of Enlargement; and *A Method for Calculating Simultaneously all the Roots of an Equation.* By Emory McClintock. Reprinted from the *American Journal of Mathematics*, vol. XVII, Nos. 1 and 2.

Dr. McClintock read his paper on "Theorems in the Calculus of Enlargement" before the American Mathematical Society, August 14, 1894, where it was received with much appreciation. The paper on "A Method for Calculating Simultaneously all the Roots of an Equation" was read before the American Mathematical Society, August 14 and October 27, 1894. This paper presents new results in the main line of analysis which should be immediately incorporated in our text-books. Any one interested in these papers should write to Dr. McClintock, Columbia College, N. Y., for a copy. B.F.F.

The Review of Reviews: An International Illustrated Monthly Magazine. Edited by Albert Shaw. Price, $2.50 per year. Single Number, 25 cents. The Review of Reviews Co. New York City.

In the *Review of Reviews* for April the editor discusses recent political events, especially the doings of the Fifty-third Congress, the appointment of delegates to an international monetary conference, the election of U. S. senators by various state legislatures, the deadlock in Delaware, the constitutional convention in Utah, the arguments before the Supreme Court on the constitutionality of the income tax, the change in the administration of the Post Office Department, and other incidents of the month under review. Persons who can only afford to take one of the leading Literary Magazines of the World and desire to know what is going on all over the world should subscribe for *The Review of Reviews.* B.F.F.

The Cosmopolitan: an Illustrated Monthly Magazine. Edited by

John Brisben Walker and Arthur Sherburne Hardy. Price, $1.50 per year. Single Number, 15 cents.

The following are some of the leading articles in the April Number: The Nymph of the Attitudes, by Mrs. Robert P. Porter; The Late Returning, by Gertrude Hall; English Wood-Notes, by James Lane Allen; The Krakatoa Eruption, by Jean T. VanGestel; and The Story of a Thousand, by Albion W. Tourgee. The description of Krakatoa Eruption, by an eyewitness, is full of thrilling interest from beginning to end. The Cosmopolitan is one of the very best Magazines published in America, and its price is so reasonable that it is easily within the reach of all. You might be charged more than 15 cents for such a Number but could it contain better materal? See our offer in the December No. of the MONTHLY. B.F.F.

The Mathematical Gazette: a Terminal Journal for Students and Teachers. Edited by E. M. Langley, M. A., Published by Macmillan & Co., London and New York. No. 4, February, 1895, 4to, pp. 25-36. Price, One Shilling Net. Subscription for 1895, 7s. 6d.

The February Number contains an excellent paper on "Mathematics for Astronomy and Navagation", by T. Wilson. The paper on "Algebra in Schools", by G. Heppel, offers some good suggestions. The paper suggests that $\sqrt{a}$ should be considered as having one and not two values. Without offering a criticism to this statement, we believe that it is best to consider $\sqrt{a}$ as having always two values, though both values may not be admissable in the same equation. B.F.F.

The Mathematical Magazine: a Journal of Elementary and Higher Mathematics. Edited and Published by Artemas Martin, M.A.,Ph.D.,LL.D., Washington, D. C. Terms: $1.00 in advance for Four Numbers.

The January, 1895, Number contains the following papers: About Cube Numbers Whose Sum is a Cube, by Dr. Martin; The United States Bond Problem, by Theodore L. DeLand; On the Celebrated Cattle Problem of Archimedes, by A. H. Bell. The solutions of six problems are published and ten new problems are proposed for solution. The appearance and typographical execution of this Number is, as usual, first class. B.F.F.